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Nonisomorphic Hadamard Designs

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A block $\mathscr{b}$ of a Hadamard design is called a good block if the symmetric difference $\mathscr{b} + \mathscr{b}_1$ is also a block for all nonparallel blocks $\mathscr{b}_1$. The isomorphism classes of such designs having a good block are shown to be related to a double coset decomposition of a symmetric group. As an example, over one million mutually nonisomorphic 3-(32, 16, 7) designs of a certain type are constructed.

Equivalence of Hadamard matrices is described in terms of designs and it is shown that nonisomorphic designs may arise from the same matrix.

INTRODUCTION

It is well known that there is, up to isomorphism, a unique design with parameters 3-(8, 4, 1) and there is also a unique 3-(12, 6, 2) design. It has been shown that there are exactly five nonisomorphic 3-(16, 8, 3) designs [3] and exactly three nonisomorphic 3-(20, 10, 4) designs [5]. A 3-($4\lambda + 4, 2\lambda + 2, \lambda$) design is called a Hadamard design with parameter λ; there is a close connection between such designs and Hadamard matrices [10, Chap. 8], [2, Chap. 2]. Hadamard designs and matrices have been extensively studied and the problem of determining the number of nonisomorphic Hadamard designs with a given parameter has been considered [1].

Here we find a lower bound for the number of nonisomorphic Hadamard designs which may be constructed by a method due to Todd [11]; as an illustration, over one million nonisomorphic 3-(32, 16, 7) designs of a certain type are exhibited. We also investigate the connection between nonisomorphic Hadamard designs and inequivalent Hadamard matrices and give an example of two equivalent but nonisomorphic 3-(24, 12, 5) designs.

1. Good Block Classes

Let $\mathscr{D}$ be a Hadamard design with parameter λ. If ℓ is a block of $\mathscr{D}$ then the complement ℓ' of ℓ is also a block of $\mathscr{D}$. A pair of complementary blocks is called a block class and we write $\bar{\ell} = \{\ell, \ell'\}$. There are $8\lambda + 6$ blocks and $4\lambda + 3$ block classes in $\mathscr{D}$. We denote by $\{\mathscr{D}\}$ the point-set of the design $\mathscr{D}$.

If $\mathscr{D}^1$ and $\mathscr{D}^2$ are Hadamard designs with the same parameter λ, let $\mathscr{C}^i$ denote the set of block classes of $\mathscr{D}^i$ and let $f\colon \mathscr{C}^1 \to \mathscr{C}^2$ be any bijection. Following Todd [11] we may construct the design $\mathscr{D}^1 +^f \mathscr{D}^2$: the point-set of $\mathscr{D}^1 +^f \mathscr{D}^2$ is the disjoint union of the point-sets of $\mathscr{D}^1$ and $\mathscr{D}^2$; the blocks of $\mathscr{D}^1 +^f \mathscr{D}^2$ consist of all the disjoint unions $\ell^1 + \ell^2$, where ℓ^i is a block of $\mathscr{D}^i$ and $\overline{\ell^1}f = \overline{\ell^2}$, together with $\{\mathscr{D}^1\}$ and $\{\mathscr{D}^2\}$. It is easily verified that $\mathscr{D}^1 +^f \mathscr{D}^2$ is a Hadamard design with parameter $2\lambda + 1$.

A block ℓ of a Hadamard design $\mathscr{D}$ is called a good block if the symmetric difference $\ell + \ell_1$ is a block of $\mathscr{D}$, for all blocks $\ell_1 \notin \bar{\ell}$. If ℓ is a good block then so is ℓ' and in this case $\bar{\ell}$ is called a good block class of $\mathscr{D}$. For example, in the Hadamard design formed by the points and hyperplanes of n-dimensional affine space over GF(2), $n \geqslant 3$, every block class is good and indeed these are the only Hadamard designs with this property [7]. The block class $\{\{\mathscr{D}^1\}, \{\mathscr{D}^2\}\}$ is a good block class of $\mathscr{D}^1 +^f \mathscr{D}^2$ and the derived design on the block $\{\mathscr{D}^i\}$ is isomorphic to $\mathscr{D}^i$; conversely if $\mathscr{D}$ is a Hadamard design with a good block class, then $\mathscr{D}$ is isomorphic to a design of the form $\mathscr{D}^1 +^f \mathscr{D}^2$, where $\mathscr{D}^1$ and $\mathscr{D}^2$ are the derived designs on the two blocks of the given good block class. If ℓ, ℓ_1, and $\ell + \ell_1$ are blocks of $\mathscr{D}$ we write $\bar{\ell} + \bar{\ell}_1 = \overline{\ell + \ell_1}$; in particular if $\bar{\ell}$ and $\bar{\ell}_1$ are both good then so is $\bar{\ell} + \bar{\ell}_1$ and the good block classes with this addition form the nonzero elements of an elementary abelian 2-group [7].

We now investigate the existence of further good block classes in a design of the form $\mathscr{D}^1 +^f \mathscr{D}^2$.

Lemma 1. *Let $\overline{\ell^1 + \ell^2}$ be a block class of $\mathscr{D}^1 +^f \mathscr{D}^2$. Then $\overline{\ell^1 + \ell^2}$ is a good block class of $\mathscr{D}^1 +^f \mathscr{D}^2$ if and only if $\overline{\ell^i}$ is a good block class of $\mathscr{D}^i$, $i = 1, 2$ and $(\overline{\ell^1 + \ell^3})f = \overline{\ell^1}f + \overline{\ell^3}f$ for all blocks ℓ^3 of $\mathscr{D}^1$ with $\ell^3 \notin \overline{\ell^1}$.*

Proof. Suppose $\ell^1 + \ell^2$ is a good block of $\mathscr{D}^1 +^f \mathscr{D}^2$. Let ℓ^3 denote a block of $\mathscr{D}^1$ with $\ell^3 \notin \overline{\ell^1}$. Let $\ell^4 \in \overline{\ell^3}f$, then $\ell^3 + \ell^4$ is a block of $\mathscr{D}^1 +^f \mathscr{D}^2$ which does not belong to $\overline{\ell^1 + \ell^2}$. Therefore $\ell^1 + \ell^2 + \ell^3 + \ell^4 = (\ell^1 + \ell^3) + (\ell^2 + \ell^4)$ is also a block of $\mathscr{D}^1 +^f \mathscr{D}^2$. It follows that $\ell^1 + \ell^3$ is a block of $\mathscr{D}^1$, and hence $\overline{\ell^1}$ is a good block class of $\mathscr{D}^1$. Similarly $\overline{\ell^2}$ is a good block class of $\mathscr{D}^2$. Further $\overline{\ell^1 + \ell^3}f = \overline{\ell^2 + \ell^4} = \overline{\ell^2} + \overline{\ell^4} = \overline{\ell^1}f + \overline{\ell^3}f$. The reverse implication is equally straightforward to verify.

(We note that the statement immediately before [8, Corollary 3] is incorrect.)

COROLLARY 1. *If one of* $\mathscr{D}^1$ *and* $\mathscr{D}^2$ *has no good block classes, then* $\mathscr{D}^1 +^f \mathscr{D}^2$ *has a unique good block class.*

Proof. This is a direct consequence of the previous lemma. We remark that the image of a good block class of $\mathscr{D}$ under an automorphism of $\mathscr{D}$ is again a good block class of $\mathscr{D}$. Therefore, under the conditions of the above corollary, the full automorphism group of $\mathscr{D}^1 +^f \mathscr{D}^2$ fixes the unique good block class and if, in addition, $\mathscr{D}^1$ and $\mathscr{D}^2$ are nonisomorphic, this group fixes $\{\mathscr{D}^1\}$ and $\{\mathscr{D}^2\}$ [8].

2. SPECIALLY ISOMORPHIC DESIGNS

In this section we consider Hadamard designs having a distinguished good block and isomorphisms of such designs which preserve this good block. Specifically, the designs $\mathscr{D}^1 +^f \mathscr{D}^2$ and $\mathscr{D}^1 +^g \mathscr{D}^2$ are called specially isomorphic if there exists an isomorphism $\alpha\colon \mathscr{D}^1 +^f \mathscr{D}^2 \to \mathscr{D}^1 +^g \mathscr{D}^2$ satisfying $\{\mathscr{D}^1\}^\alpha = \{\mathscr{D}^1\}$.

THEOREM 1. *Let* $\mathscr{D}^1$ *and* $\mathscr{D}^2$ *be Hadamard designs with the same parameter. Let* G^i *denote the full automorphism group of* $\mathscr{D}^i$ *and denote by* $\overline{G^i}$ *the representation of* G^i *on the block classes of* $\mathscr{D}^i$. *Then* $\mathscr{D}^1 +^f \mathscr{D}^2$ *and* $\mathscr{D}^1 +^g \mathscr{D}^2$ *are specially isomorphic if and only if* $\overline{G^1}\, f\, \overline{G^2} = \overline{G^1}\, g\, \overline{G^2}$.

Proof. Suppose first that $\mathscr{D}^1 +^f \mathscr{D}^2$ and $\mathscr{D}^1 +^g \mathscr{D}^2$ are specially isomorphic. If α is an isomorphism between these designs with $\{\mathscr{D}^1\}^\alpha = \{\mathscr{D}^1\}$, then $\{\mathscr{D}^2\}^\alpha = \{\mathscr{D}^2\}$. Let α_i denote the restriction of α to the derived design defined by the block $\{\mathscr{D}^i\}$. This derived design is isomorphic $\mathscr{D}^i$, so $\alpha_i \in G^i$. If $\mathscr{b}^1$ is a block of $\mathscr{D}^1$ and $\mathscr{b}^1 + \mathscr{b}^2$ is a block of $\mathscr{D}^1 +^f \mathscr{D}^2$, we have that $(\mathscr{b}^1 + \mathscr{b}^2)^\alpha = (\mathscr{b}^1)^{\alpha_1} + (\mathscr{b}^2)^{\alpha_2}$ is a block of $\mathscr{D}^1 +^g \mathscr{D}^2$. Therefore $\overline{(\mathscr{b}^1)^{\alpha_1}}{}^g = \overline{(\mathscr{b}^2)^{\alpha_2}}$ and $\overline{\mathscr{b}^1}{}^f = \overline{\mathscr{b}^2}$. This yields $(\overline{\mathscr{b}^1})^{\bar\alpha_1 g} = (\overline{\mathscr{b}^1})^{f\bar\alpha_2}$ as $\overline{(\mathscr{b}^1)^{\alpha_1}} = (\overline{\mathscr{b}^1})^{\bar\alpha_1}$ and $\overline{(\mathscr{b}^2)^{\alpha_2}} = (\overline{\mathscr{b}^2})^{\bar\alpha_2}$. Hence $\bar\alpha_1 g = f\bar\alpha_2$ and so $\overline{G^1}\, f\, \overline{G^2} = \overline{G^1}\, g\, \overline{G^2}$.

Conversely if $\overline{G^1}\, f\, \overline{G^2} = \overline{G^1}\, g\, \overline{G^2}$ then there exist automorphisms $\alpha_i \in G^i$ satisfying $\bar\alpha_1 g = f\bar\alpha_2$. Let α be that mapping: $\mathscr{D}^1 +^f \mathscr{D}^2 \to \mathscr{D}^1 +^g \mathscr{D}^2$, the restriction of which to the derived design defined by $\{\mathscr{D}^i\}$ is α_i. Then α is a special isomorphism. This establishes the theorem.

If λ is the common parameter of the Hadamard designs $\mathscr{D}^i$, let $\Omega = \{1, 2,..., 4\lambda + 3\}$ and identify $\mathscr{C}^1 = \mathscr{C}^2 = \Omega$; in other words we number the block classes in $\mathscr{D}^1$ and $\mathscr{D}^2$. The group $\overline{G^i}$ may be regarded as a subgroup of the symmetric group S_Ω and f, g elements of S_Ω.

COROLLARY 2. *The special isomorphism classes of designs* $\mathscr{D}^1 +^f \mathscr{D}^2$ *are in one to one correspondence with the double cosets* $\overline{G^1} f \overline{G^2}$ *in* S_Ω.

Proof. The corollary is merely an alternative description of Theorem 1.

COROLLARY 3. *Let* $\mathscr{D}^1$ *and* $\mathscr{D}^2$ *be Hadamard designs with common parameter* λ. *There are at least* $(4\lambda + 3)!\,(|\overline{G^1}|\,|\overline{G^2}|)^{-1}$ *special isomorphism classes of designs* $\mathscr{D}^1 +^f \mathscr{D}^2$.

Proof. Each double coset contains at most $|\overline{G^1}|\,|\overline{G^2}|$ elements.

COROLLARY 4. *Let* $\mathscr{D}^1$ *and* $\mathscr{D}^2$ *be nonisomorphic Hadamard designs with common parameter* λ. *If one of* $\mathscr{D}^1$ *and* $\mathscr{D}^2$ *has no good blocks then there are at least* $(4\lambda + 3)!\,(|\overline{G^1}|\,|\overline{G^2}|)^{-1}$ *mutually nonisomorphic designs of the form* $\mathscr{D}^1 +^f \mathscr{D}^2$.

Proof. By Corollary 1 and the remark following it, every isomorphism between two designs of the form $\mathscr{D}^1 +^f \mathscr{D}^2$ must be special. The result now follows from Corollary 3.

3. NONISOMORPHIC HADAMARD DESIGNS WITH PARAMETER 7

In this section we derive a crude lower bound for the number of nonisomorphic Hadamard designs with parameter 7 having a unique good block class with given associated derived designs.

There are five mutually nonisomorphic 2-(15, 7, 3) designs [11]. Each of these designs can be uniquely extended to a Hadamard design with parameter 3. These Hadamard designs each admit point-transitive automorphism groups and so are themselves mutually nonisomorphic. In particular Todd constructs two 2-(15, 7, 3) designs which he denotes by A^8C^7 and B^8C^6D [11]. We use the extensions of these designs denoting them by $\mathscr{D}^1$ and $\mathscr{D}^2$ respectively. $\mathscr{D}^1$ has no good blocks and $\mathscr{D}^2$ has a unique good block class. Therefore $\mathscr{D}^1$ and $\mathscr{D}^2$ satisfy the conditions of Corollary 4.

Using the notation of [11], the full automorphism group of A^8C^7 is of order 168; this group fixes the point a and is transitive on the remaining points. The extension $\mathscr{D}^1$ admits the involution $(ac)(bt)(df) \times (eg)(hr)(ks)(lp)(mq)$ where t denotes the added point. Therefore G^1 has order 16×168. Also $\mathscr{D}^1$ admits a unique translation $\tau \neq 1$; τ has point permutation $(ta)(bc)(de)(fg)(hk)(lm)(pq)(rs)$ and so $|\overline{G^1}| = 8 \times 168$. On the other hand B^8C^6D has a full automorphism group of order 96 and the orbits of this group are $\{a\}$, $\{bcdefg\}$, $\{hklmpqrs\}$. The extension $\mathscr{D}^2$

admits the involution $(al)(bh)(ck)(dq)(ep)(fr)(gs)(tm)$ and so $| G^2 | = 16 \times 96$. As above, $\mathscr{D}^2$ admits the unique nonidentity tranlation τ; hence $| \overline{G^2} | = 8 \times 96$.

THEOREM 2. *Let $\mathscr{D}^1$ and $\mathscr{D}^2$ be the Hadamard designs with paramater 3 described above. There are at least* 1,266,891 *mutually nonisomorphic designs of the form $\mathscr{D}^1 +^f \mathscr{D}^2$.*

Proof. This is a direct consequence of Corollary 4 substituting the values $\lambda = 3$, $| \overline{G^1} | = 8 \times 168$, $| \overline{G^2} | = 8 \times 96$.

4. EQUIVALENCE OF HADAMARD MATRICES

Let H and K be Hadamard matrices. H and K are called equivalent if there exist generalized permutation matrices X and Y such that $X'HY = K$; the group $G(H)$, of all pairs (X, Y) of generalized permutation matrices satisfying $X'HY = H$, is called the automorphism group of H [4].

If n is the order of the Hadamard matrix $H = (a_{il})$ then n designs $\mathscr{D}_j(H)$ may be formed as follows $(1 \leqslant j \leqslant n)$: the points of $\mathscr{D}_j(H)$ are the symbols P_i $(1 \leqslant i \leqslant n)$ and the blocks of $\mathscr{D}_j(H)$ are the subsets $b(+l)$ and $b(-l)$, where $b(+l) = \{P_i \mid a_{il} = a_{ij}\}$, $b(-l) = \{P_i \mid a_{il} = -a_{ij}\}$, for $l \neq j$, $(1 \leqslant l \leqslant n)$. It is easy to check that $\mathscr{D}_j(H)$ is a Hadamard design with parameter $(n/4) - 1$ provided $n \geqslant 8$.

We now discuss the connection between isomorphisms of these Hadamard designs and equivalences of the underlying matrices. Suppose that α is an isomorphism: $\mathscr{D}_j(H) \to \mathscr{D}_k(K)$. Since α preserves block classes, $\{b(+l)^\alpha, b(-l)^\alpha\} = \{b(+m), b(-m)\}$ and we define $\tilde{\alpha}$ by setting $l^{\tilde{\alpha}} = m$; we extend $\tilde{\alpha}$ to an element of S_n by defining $j^{\tilde{\alpha}} = k$. Also let $\bar{\alpha}$ denote the point permutation determined by α, so $i^{\bar{\alpha}} = i'$ if $P_i^\alpha = P_{i'}$. Finally, if X is a generalized permutation matrix we use $\overline{X}$ to denote the permutation determined by X.

THEOREM 3. *Let H, K and α be as above. There exist generalized permutation matrices X and Y satisfying $X'HY = K$ and $\overline{X} = \bar{\alpha}$, $\overline{Y} = \tilde{\alpha}$; further, there is exactly one other pair of generalized permutation matrices with this property, namely $(-X, -Y)$.*

Proof. Let $H = (a_{il})$ and $K = (b_{il})$. Define $X = (x_{il})$ by $x_{il} = 0$ if $l \neq i^{\bar{\alpha}}$, and $x_{il} = \pm 1$ according as $a_{ij} = \pm b_{lk}$ if $l = i^{\bar{\alpha}}$. Clearly $\overline{X} = \bar{\alpha}$. Let $Y = (y_{lm})$ where $y_{lm} = 0$ if $m \neq l^{\tilde{\alpha}}$, $y_{jk} = 1$, and $y_{lm} = \pm 1$ according as $b(+l)^\alpha = b(\pm m)$ if $m = l^{\tilde{\alpha}}$, $l \neq j$. Clearly $\overline{Y} = \tilde{\alpha}$.

The point P_i is incident with the block $b(a_{il}a_{ij}l)$ in $\mathscr{D}_j(H)$. Let $i' = i^{\tilde{\alpha}}$ and $m = l^{\bar{\alpha}}$. As α preserves incidence, $P_{i'}$ is incident with $b(a_{il}a_{ij}l) = b(a_{il}a_{ij}y_{lm}m)$ in $\mathscr{D}_k(K)$. Therefore $b_{i'm} = a_{il}a_{ij}y_{lm}b_{i'k} = x_{ii'}a_{il}y_{lm} =$ $(i'm)$-element of $X'HY$, as $x_{ii'} = a_{ij}b_{i'k}$. Also $b_{i'k} = x_{ii'}a_{ij}y_{jk} = (i', k)$-element of $X'HY$. Therefore $K = X'HY$.

The last statement of the theorem follows immediately from the fact that $(\pm I, \pm I)$ are the only automorphisms of H which induce the identity permutation on the rows and columns of H [4].

THEOREM 4. *Let H and K be Hadamard matrices and X and Y generalized permutation matrices such that $X'HY = K$. Then there exists a unique isomorphism $\alpha\colon \mathscr{D}_j(H) \to \mathscr{D}_k(K)$ satisfying $\tilde{\alpha} = \overline{X}$, $\bar{\alpha} = \overline{Y}$, where k is the image of j under $\overline{Y}$.*

Proof. Clearly there is at most one isomorphism satisfying the conditions of the theorem. If $l \neq j$, we define $b(\pm l)^{\alpha} = b(\pm y_{lm}y_{jk}m)$ where $y_{lm} \neq 0$. As P_i is incident with $b(a_{il}a_{ij}l)$, it suffices to show that $P_{i'}$ is incident with $b(a_{il}a_{ij}l)$, where $P_i^{\alpha} = P_{i'}$ if $x_{ii'} \neq 0$. Equivalently we must show $b_{i'k} = a_{il}a_{ij}y_{lm}y_{jk}b_{i'm}$. However, comparing the (i', k)-elements and the (i', m)-elements of the matrix equation $X'HY = K$, we see that this relation does hold. Therefore α is an isomorphism satisfying $\tilde{\alpha} = \overline{X}$, $\bar{\alpha} = \overline{Y}$.

COROLLARY 5. *The Hadamard matrices H and K are equivalent if and only if there exist j and k such that $\mathscr{D}_j(H)$ and $\mathscr{D}_k(K)$ are isomorphic.*

Proof. This is a direct consequence of Theorem 3 and Theorem 4.

COROLLARY 6. *Let H be a Hadamard matrix and let $G(H)$ denote the representation of $G(H)$ on the columns of H. The number of orbits of $G(H)$ is equal to the number of isomorphism classes of designs of the form $\mathscr{D}_j(H)$.*

Proof. Set $H = K$ in Theorems 3 and 4. Then $\mathscr{D}_j(H)$ and $\mathscr{D}_k(K)$ are isomorphic if and only if j and k belong to the same orbit of $G(H)$.

COROLLARY 7.[1] *Let $\mathscr{D}^1$ and $\mathscr{D}^2$ be the Hadamard designs defined in the previous section. The Hadamard matrices H of order 32 such that $\mathscr{D}_j(H)$ is isomorphic $\mathscr{D}^1 +^f \mathscr{D}^2$, for some j and f, belong to at least 39,591 different equivalence classes.*

[1] Wallis [12] has shown that the Hadamard matrices of order *32* fall into *11* Z-equivalence classes; this directly implies that there exist at least *11* equivalence classes of such matrices.

Proof. First we recall that every Hadamard design can be expressed in the form $\mathscr{D}_j(H)$; the most familiar method of effecting this has $j = 1$ and $a_{i1} = 1$ $(1 \leqslant i \leqslant n)$ [2]. By Corollary 5, inequivalent Hadamard matrices give rise to disjoint sets of isomorphism types of designs $\mathscr{D}_j(H)$. Further, an equivalence class of Hadamard matrices of order n gives rise to at most n distinct isomorphism types. Therefore by Theorem 2 there are at least 39,591 mutually nonequivalent Hadamard matrices of order 32 having the stated property.

5. Equivalent Hadamard Designs

Every Hadamard design can be represented in the form $\mathscr{D}_j(H)$, where H is a suitable Hadamard matrix. By Corollary 5 the equivalence class of H is determined by the given Hadamard design. We call two Hadamard designs $\mathscr{D}_j(H)$ and $\mathscr{D}_k(K)$ equivalent if H and K are equivalent. Here we show how to construct all the Hadamard designs which are equivalent to a given Hadamard design and we exhibit an example of two equivalent but nonisomorphic Hadamard designs.

Let $\bar{\mathscr{E}}$ be a block class of the Hadamard design $\mathscr{D}$. The design $\mathscr{D}_{\bar{\mathscr{E}}}$ is defined to have the same point-set as $\mathscr{D}$, but the blocks of $\mathscr{D}_{\bar{\mathscr{E}}}$ are $\mathscr{E}$, $\mathscr{E}'$ and the sets $\mathscr{E} + \mathscr{E}_1$ for all blocks $\mathscr{E}_1$ of $\mathscr{D}$ with $\mathscr{E}_1 \notin \bar{\mathscr{E}}$. It is straightforward to verify that $\mathscr{D}_{\bar{\mathscr{E}}}$ is a Hadamard design which is equivalent to $\mathscr{D}$; in fact if $\mathscr{D} = \mathscr{D}_j(H)$ and $\bar{\mathscr{E}}$ corresponds to column $k \neq j$ of H, then $\mathscr{D}_{\bar{\mathscr{E}}} = \mathscr{D}_k(H)$. Conversely, given a design which is equivalent to $\mathscr{D}$, that is, of the form $\mathscr{D}_l(K)$, where $K = X'HY$, then this design is isomorphic $\mathscr{D}_k(H)$ by Theorem 4, l being the image of k under $\bar{Y}$. Therefore every Hadamard design which is equivalent to $\mathscr{D}$ is isomorphic either to $\mathscr{D}$ or to $\mathscr{D}_{\bar{\mathscr{E}}}$, for some block class $\bar{\mathscr{E}}$ of $\mathscr{D}$.

There is a natural correspondence between the block classes of $\mathscr{D}$ and those of $\mathscr{D}_{\bar{\mathscr{E}}}$, namely $\bar{\mathscr{E}}_1$ corresponds to $\overline{\mathscr{E} + \mathscr{E}_1}$ if $\bar{\mathscr{E}}_1 \neq \bar{\mathscr{E}}$ and $\bar{\mathscr{E}}$ corresponds to $\bar{\mathscr{E}}$. With this identification we have the following formula.

Lemma 2. $(\mathscr{D}^1 +^f \mathscr{D}^2)_{\overline{\mathscr{E}^1+\mathscr{E}^2}} = (\overset{1}{\mathscr{D}}_{\bar{\mathscr{E}}^1}) +^f (\overset{2}{\mathscr{D}}_{\bar{\mathscr{E}}^2})$, *where* $\mathscr{D}^1$ *and* $\mathscr{D}^2$ *are Hadamard designs with the same parameter and* $\overline{\mathscr{E}^1}f = \overline{\mathscr{E}^2}$.

Proof. The point-sets of the two designs are identical. Also both designs have the same blocks, namely $\{\mathscr{D}^1\}$, $\{\mathscr{D}^2\}$, $\mathscr{E}^1 + \mathscr{E}^2$, $(\mathscr{E}^1)' + (\mathscr{E}^2)'$, $(\mathscr{E}^1)' + \mathscr{E}^2$, $\mathscr{E}^1 + (\mathscr{E}^2)'$ together with the subsets $\mathscr{E}^1 + \mathscr{E}^2 + \mathscr{E}^3 + \mathscr{E}^4$ for all blocks $\mathscr{E}^3 \notin \overline{\mathscr{E}^1}$ and $\mathscr{E}^4 \in \overline{\mathscr{E}^3}f$.

Let H denote a Hadamard matrix of order 12. There is, up to equivalence, only one such matrix and $G(H)$ is permutation isomorphic to the sharply

5-transitive Mathieu group M_{12} [4]. Let $\mathscr{D} = \mathscr{D}_1(H)$, then the block classes of $\mathscr{D}$ correspond to columns 2 to 12 of H and will be labeled accordingly. Let f denote a transposition of two of the block classes of $\mathscr{D}$, so $f = (u, v)$ where $2 \leqslant u < v \leqslant 12$.

THEOREM 5. *$\mathscr{D} +^f \mathscr{D}$ and $(\mathscr{D}_u) +^f (\mathscr{D}_v)$ are equivalent nonisomorphic Hadamard designs with parameter* 5.

Proof. By Lemma 2 and the preceding discussion, $\mathscr{D} +^f \mathscr{D}$ and $(\mathscr{D}_u) +^f (\mathscr{D}_v)$ are equivalent. By Theorem 4 there exist isomorphisms $\alpha: \mathscr{D}_u \to \mathscr{D}$, $\beta: \mathscr{D}_v \to \mathscr{D}$. Further $\bar{\alpha}, \bar{\beta} \in \overline{G(H)}$ and $u^{\bar{\alpha}} = 1$, $v^{\bar{\beta}} = 1$. Also $(\mathscr{D}_u) +^f (\mathscr{D}_v) \approx \mathscr{D} +^g \mathscr{D}$, where $g = \bar{\alpha}^{-1} f \bar{\beta}$. Now $\mathscr{D} +^f \mathscr{D}$ has an automorphism interchanging the blocks of the unique good block class [8, Theorem 3], for it suffices to take $\alpha_1 = \alpha_2 =$ identity as $f^2 =$ identity. Therefore $\mathscr{D} +^f \mathscr{D}$ and $\mathscr{D} +^g \mathscr{D}$ are isomorphic if and only if they are specially isomorphic. Suppose this to be the case. By Theorem 1 there exist γ, $\delta \in \operatorname{Aut} \mathscr{D}$ with $\bar{\gamma} f = g \bar{\delta}$. Therefore $\bar{\alpha}\bar{\gamma} f = f \bar{\beta}\bar{\delta}$. This implies $\bar{\alpha}\bar{\gamma} = \bar{\beta}\bar{\delta}$ since these products belong to the sharply 5-transitive group M_{12} and they agree on at least eight symbols. As $\bar{\gamma}$ and $\bar{\delta}$ both fix the symbol 1, so also does $(\bar{\beta})^{-1} \bar{\alpha}$. But this implies $u = v$. This contradiction establishes Theorem 5.

Finally we remark that if H is of Sylvester type or of Paley type [9], or if the order of H is at most 20, then $G(H)$ is transitive [4, 6]. Hence 24 is the smallest order for which there exist nonisomorphic equivalent Hadamard designs.

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